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Blocking and Dimer Processes on the Cayley Tree

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Abstract We show that the equilibrium distribution for the dimer process on the finite Cayley tree tends to a translation invariant limit as the size of the tree tends to infinity. The same is true for the blocking process except when there is a phase transition, in which case there are two limits, each a one-step translation of the other. We also find correlations for occupation probabilities.

Keywords Particle system · Equilibrium distribution · Cayley tree

1 Introduction

In this paper we consider two types of interacting particle processes on the finite Cayley tree, where each vertex is either occupied (1) or unoccupied (0). In the first, called the blocking process, particles leave ($1 \to 0$) at a constant rate, while the rate at which they arrive ($0 \to 1$) depends on the number of occupied neighbours. One extreme version of the blocking process is known as the hardcore model, where all neighbouring sites have to be unoccupied for the particle to stick. In the dimer process particles can only move in pairs, and thus may arrive at or leave neighbouring pairs of sites at a constant rate.

There are well known models similar to those described above, except in that particles can only arrive but not leave. These processes necessarily terminate and are classified under the heading of Random Sequential Adsorption (RSA). Some of the physics literature for RSA is summarised in [2] and some of the mathematical in [8]. Renyi's classical parking problem is a continuous version of RSA in one dimension.

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If the rates of arrival are those of Glauber dynamics, then it is well-known that the blocking process possesses an Ising-type equilibrium which exhibits a phase transition [3, 7]. The hardcore version is noted in [4] and the general case is treated in Chap. 4 of [1]. An extension to irregular trees is given in [6].

The conditions for a phase transition are given in Theorem 5. It states that if the basic rate of arrival is large enough, and the blocking by neighbouring particles strong enough, a phase transition will occur. The method used is similar to that in [1], but our intention is to interpret the phase transition in terms of the blocking model, and our formulae are not readily extracted from his treatment, which uses the parameters of the conventional Ising model. For the dimer model we show there is no phase transition.

The processes will be defined on the finite Cayley tree $T_k^{(n)}$, a tree radius n with the centre having $k+1$ edges, and all other vertices or sites having $k+1$ edges except for those that are distance n from the centre, which have one edge. The state-space is the set X of functions, also called configurations, assigning the value 0 or 1 to each vertex. Since the processes are Markov and the state-spaces are finite they converge to a unique equilibrium distribution designated by $P^{(n)}$. For a given a set of sites A of $T_k^{(n)}$ and a configuration $\eta \in X$, η^A denotes the restriction of η to A and a pattern $\pi(A)$ is defined by a set of values 0, 1 assigned to the sites in A. In other words, a pattern $\pi(A)$ can be identified with a function in $\{0,1\}^A$. The probability of a pattern with respect to the equilibrium distribution is obtained as the sum of the probabilities of compatible configurations. That is,

$$P^{(n)}(\pi(A)) = \sum P^{(n)}(\eta) I_{\{\eta^A = \pi(A)\}}.$$

For a pattern $\pi(A)$ on $T_k^{(n)}$ we let $n \to \infty$ keeping the position of the pattern relative to the root. We show that, when there is not a phase transition, $P^{(n)}(\pi(A))$ tends to a translation invariant limit and that, when there is a phase transition, there are different limits as $n \to \infty$ through odd and even values, but that these limits are one-step translations of each other.

We further calculate correlations for both processes showing that they decrease geometrically, in contrast to RSA where the decrease is as the reciprocal factorial of the distance.

The rest of the paper takes the following form. Conditions for phase transition in the blocking process are given in Sect. 2.2, occupation probabilities and convergence in Sects. 2.3–2.5, correlations in Sect. 2.7. Section 3 covers the dimer process with occupation probabilities and convergence in Sects. 3.1–3.3 and correlations in Sect. 3.4.

2 The Blocking Process

2.1 The Blocking Process on Infinite Isotropic Bipartite Graphs

The vertices of a bipartite graph can be put into two disjoint sets each of which has its neighbours in the other. We shall call these two sets *odd* and *even*. The blocking process η_t is defined so that the arrival rate at an empty site is a decreasing function of the number of occupied neighbours. Consider the transformation β on X, defined by $\beta(\eta)(x) = \eta(x)$, for x even and $\beta(\eta)(x) = 1 - \eta(x)$, for x odd. The transformed process $\xi_t = \beta(\eta_t)$ is *attractive* (see Definition 2.3 in Chap. II of [5]), since the rate from 0 to 1 is a non-decreasing function of the number of occupied neighbours and the rate from 1 to 0 non-increasing. Note that these rates are not the same for the odd and even sites.

Let δ_1 be the configuration *all occupied*, δ_0 the configuration *all empty* and ξ_t^δ the process ξ_t starting from configuration δ. Since ξ_t is attractive it follows, from Theorem 2.3 in

Chap. III of [5], that $\xi_t^{\delta_1}$ converges to the upper invariant measure ν^+, and $\xi_t^{\delta_0}$ to the lower invariant measure ν^-. But δ_1 in the ξ_t process corresponds to the configuration $\beta(\delta_1)$ where the even set starts all occupied and the odd empty for the η_t process. Then $\eta_t^{\beta(\delta_1)}$ must converge to the measure $\mu^+(\cdot) = \nu^+(\beta(\cdot))$. On the other hand, δ_0 in the ξ_t process corresponds to the configuration $\beta(\delta_0)$ where the odd set starts all occupied and the even empty for the η_t process, which on trees and Z^d is clearly a one-step translation of $\beta(\delta_1)$. Thus the limiting measure $\mu^-(\cdot) = \nu^-(\beta(\cdot))$ is also a one-step translation of μ^+, in the sense that $\mu^-(\cdot) = \mu^+(\tau(\cdot))$, where τ is the one-step translation. A phase transition occurs when the occupation probabilities for the odd and even sites are not equal. We shall see this result reflected in our analysis of finite trees.

Theorem 1 *Let η_t be a blocking process on either a regular infinite tree or Z^d in which the rate of arrival at an empty site is a non-increasing function of the number of occupied neighbouring sites. Then the process converges from an initial state in which the even sites are all occupied and the odd unoccupied.*

In the theorem above, if the limiting measure is not one-step translation invariant, we say that a phase transition occurs. Note that we do not require the limiting measures to be Gibbs.

2.2 The Blocking Process on Finite Trees

We shall consider a blocking process such that the rate at which particles depart ($1 \to 0$) is 1, and the rate at which they arrive ($0 \to 1$) is $\lambda\lambda_2^r$, where r is the number of occupied neighbours. Note that $\lambda_2 > 1$ means that occupied neighbours increase the chance of a particle arriving, $\lambda_2 < 1$ that they are inhibitory and $\lambda_2 = 0$ is the hardcore model. In the usual notation for interacting particle systems the flip-rate is

$$c(x, \eta) = \eta(x) + (1 - \eta(x))\lambda\lambda_2^{\sum_{y \in N_x} \eta(y)},$$

where $N_x = \{y : |y - x| = 1\}$. These are the Glauber dynamics of an Ising model and the equilibrium distribution has the probability of a configuration proportional to $\lambda^{\#\text{occupied sites}}\lambda_2^{\#\text{occupied pairs}}$, where pairs should always be understood as pairs of neighbouring sites. This is a Gibbs measure and would be written $\exp[K\sum\sigma_i\sigma_j + h\sum\sigma_i]$ in the physics literature.

Note that the connection between λ, λ_2 and K, h is not immediate because the σ_i take the values $-1, 1$.

We shall call $\lambda^{\#\text{occupied sites}}\lambda_2^{\#\text{occupied pairs}}$ the weight of a configuration, and thus the probability of a configuration in equilibrium is its weight divided by the sum of weights over all configurations, known as the partition function. The weight of a pattern is the sum of the weights of the configurations compatible with it. Likewise, the probability of a pattern is obtained by dividing the weight of the pattern by the partition function.

For a given pattern and each r_1, r_2, let $n(r_1, r_2)$ be the number of compatible configurations having r_1 occupied sites and r_2 pairs of neighbouring occupied sites. Then, the weight of the pattern is given by $\sum_{r_1,r_2} n(r_1, r_2)\lambda^{r_1}\lambda_2^{r_2}$. Thus the problem resolves into counting individual occupied sites and pairs of neighbouring occupied sites. The method used below is very similar to that in [1], Chap. 4. The treatment in [4] is the case $\lambda_2 = 0$.

Consider a finite rooted tree T and all subtrees $S_1, \ldots, S_k$ emerging from the root but not including it. Let $n_j(r_1, r_2)$ be the number of ways S_j can have r_1 occupied sites and r_2 pairs of neighbouring occupied sites, with corresponding generating function

$$G_j(s,t)=\sum_{r_1,r_2} n_j(r_1,r_2)s^{r_1}t^{r_2}.$$

Let $n(r_1,r_2)$, with generating function $G(s,t)$, be the number of ways T (the root and its subtrees) can have r_1 occupied sites and r_2 pairs of neighbouring occupied sites, with the values at the root and at its neighbours on the subtrees defined but otherwise the values at the other sites left free.

Lemma 2 *Let G and G_j be as defined above. Then*

$$G(s,t)=s^{m_0}t^{m_0m_1}\prod_{j=1}^{k} G_j(s,t),$$

where $m_0=1$ (0) if the root is occupied (unoccupied) and m_1 is the number of occupied sites neighbouring the root.

Proof Standard arguments show that $\prod_{j=1}^{k} G_j(s,t)$ gives the generating function for the numbers of occupied sites and pairs of neighbouring occupied sites, ignoring the contributions from the root itself. The root contributes $m_0=0,1$ occupied sites and there are m_0m_1 pairs of neighbouring occupied sites involving the root. □

Define $R_k^{(n)}$ to be the rooted tree with the root having k branches or edges, all other vertices or sites having $k+1$ branches except for those that are distance n from the root which have one branch. All vertices are at most distance n from the root. Let $P_n(s,t)$ be the generating function for the number of ways in which $R_k^{(n)}$ can have r_1 occupied sites and r_2 pairs of neighbouring occupied sites when the root is occupied and let $Q_n(s,t)$ be the generating function for the number of ways $R_k^{(n)}$ can have r_1 occupied sites and r_2 pairs of neighbouring occupied sites, when the root is unoccupied. Thus, the probability the root is occupied is

$$\frac{P_n}{P_n+Q_n},$$

where P_n, Q_n are to be understood as $P_n(\lambda,\lambda_2)$, $Q_n(\lambda,\lambda_2)$.

Lemma 3 *For the blocking process on $R_k^{(n)}$,*

$$P_{n+1}=\lambda(\lambda_2P_n+Q_n)^k \quad \textit{and} \quad Q_{n+1}=(P_n+Q_n)^k, \tag{1}$$

with $P_0=\lambda$ and $Q_0=1$.

Proof If there is a 1 at the root then r of its neighbours could be 1, $k-r$ could be 0. The root scores a single λ, and for each neighbouring 1 we must add a pair of 1, each scoring λ_2, so that

$$P_{n+1}=\lambda\sum_{r=0}^{k}\binom{k}{r}\lambda_2^r P_n^r Q_n^{k-r}=\lambda(\lambda_2P_n+Q_n)^k.$$

With the root unoccupied, λ, λ_2 disappear from the equation. □

Lemma 4 *Let*

$$f(x) = \frac{\lambda + x^k}{\lambda\lambda_2 + x^k} = 1 + \frac{\lambda(1-\lambda_2)}{\lambda\lambda_2 + x^k},$$

for $x > 0$, *and*

$$l_n = \frac{P_n + Q_n}{\lambda_2 P_n + Q_n}, \tag{2}$$

where P_n *and* Q_n *are defined in* (1) *and* $l_0 = f(1) = (1+\lambda)/(1+\lambda\lambda_2)$. *Then*

(i) l_n *satisfies the recursion* $l_{n+1} = f(l_n)$.
(ii) *If* $\lambda_2 < 1$, f *is strictly decreasing*, (l_{2n+1}) *converges increasingly to* l_{odd} *and* (l_{2n}) *converges decreasingly to* l_{even}. *Also, the equation* $f(x) = x$ *has a unique positive solution, to be designated by* l, *such that* $l_{odd} \le l \le l_{even}$. *If* $\lambda_2 > 1$, f *is strictly increasing and the sequence* (l_n) *converges decreasingly to* $l < 1$, *the unique solution of* $f(x) = x$.
(iii) *Let* $\lambda_c = k^k/(k-1)^{k+1}$. *For each* $\lambda > \lambda_c$ *there exists* $\lambda_2^* \in (0,1)$ *such that the blocking model with parameters* λ *and* $\lambda_2 \in (0, \lambda_2^*)$ *has* $l_{odd} < l < l_{even}$.
(iv) *If* $\lambda \le \lambda_c$, $l_{even} = l_{odd} = l$. *At* $\lambda = \lambda_c$ *and* $\lambda_2 = 0$ *the corresponding* $l_c = k/(k-1)$.

Proof (i) From (1, 2) and the definition of f we obtain

$$f(l_n) = \frac{\lambda(\lambda_2 P_n + Q_n)^k + (P_n + Q_n)^k}{\lambda\lambda_2(\lambda_2 P_n + Q_n)^k + (P_n + Q_n)^k} = l_{n+1}.$$

(ii) It is easily seen that $f'(x) < 0$ and $f(x) < f(1)$, for all $x > 1$, when $\lambda_2 < 1$. Hence, $l_0 = f(1) > f(l_0) = l_1$, $l_0 = f(1) > f(f(l_0)) = l_2$ and $l_3 = f(l_2) > f(l_0) = l_1$.

Using the inequalities for l_0, l_1 and l_2 above, we proceed inductively to show that (l_{2n}) is decreasing and (l_{2n+1}) increasing. Assume that $l_{2n+2} < l_{2n}$ then, since f is strictly decreasing, we have

$$l_{2n+3} = f(l_{2n+2}) > f(l_{2n}) = l_{2n+1},$$

$$l_{2n+4} = f(l_{2n+3}) < f(l_{2n+1}) = l_{2n+2}$$

and

$$l_{2n+5} = f(l_{2n+4}) > f(l_{2n+2}) = l_{2n+3}.$$

Therefore, $l_{2n+2} < l_{2n}$ and $l_{2n+3} > l_{2n+1}$ hold for all $n \ge 0$.

We use next a double induction argument to show that

$$l_{2n+1} < l_{2m}, \tag{3}$$

for all $m, n \ge 0$. The $n = m = 0$ case was established above. If we assume $l_{2n+1} < l_{2m}$ for arbitrary m, n, we have

$$l_{2n+1} < l_{2n+3} = f(f(l_{2n+1})) < f(f(l_{2m})) = l_{2m+2} < l_{2m},$$

since (l_{2n+1}) is increasing and (l_{2n}) decreasing. Therefore, $l_{2n+1} < l_{2m}$ holds for all $m, n \ge 0$. Convergence of (l_{2n+1}) to l_{odd} and (l_{2n}) to l_{even} follow now from the monotonicity of the sequences and inequality (3) implies $l_{odd} \le l_{even}$.

On the other hand, f is continuous, strictly decreasing and $f(x) > 1$. Then the fixed point theorem guarantees that the equation $f(x) = x$ has a unique root l such that $0 < l <$

$f(0) = \lambda_2^{-1}$. Finally, since $l < l_1$ is equivalent to $l > l_0$ but $l_1 < l_0$, we have necessarily that $l_1 \le l \le l_0$ and hence, $l_{2n+1} \le l \le l_{2n}$, for all $n \ge 0$, and $l_{odd} \le l \le l_{even}$.

When $\lambda_2 > 1$, $f'(x) > 0$ and $f(x) < f(1) < 1$, for $x < 1$. Then, $l_0 = f(1) > f(l_0) = l_1$ and $l_1 = f(l_0) > f(l_1) = l_2$. It is shown inductively that (l_n) is decreasing and hence convergent to $l = f(l)$.

(iii) We show that under the stated conditions, the blocking model with parameters λ, λ_2 has a repelling fixed point l, that is $|f'(l)| > 1$. Hence, the sequence of iterates $l_{n+1} = f(l_n)$ does not converge to l and necessarily $l_{odd} < l_{even}$. Notice first that

$$f'(x) = -\frac{kx^{k-1}\lambda(1-\lambda_2)}{(\lambda\lambda_2 + x^k)^2} = -\frac{kx^{k-1}(f(x)-1)^2}{\lambda(1-\lambda_2)} = -\frac{kx^{k-1}(f(x)-1)}{\lambda\lambda_2 + x^k}. \tag{4}$$

Given $\lambda > \lambda_c$, we solve the system of equations

$$f(l) = l, \qquad |f'(l)| = 1, \tag{5}$$

for l and λ_2 in the region $\mathcal{R} = \{(l, \lambda_2) \mid k/(k-1) < l < 1/\lambda_2, 0 < \lambda_2 < 1\}$. We verify in fact that (5) has a unique solution $(l_*, \lambda_2^*) \in \mathcal{R}$.

Using (4), the above equations can be written equivalently as

$$\lambda = \frac{l^k(l-1)}{1-\lambda_2 l}, \qquad \lambda_2 = \frac{l^k}{\lambda}(k-1-k/l), \tag{6}$$

which, after some algebraic manipulation, are found to be equivalent to

$$\lambda = l^k(k(l-1)-1), \qquad \lambda_2 = \frac{k-1-k/l}{k(l-1)-1}. \tag{7}$$

Notice that $l^k(k(l-1)-1)$ is increasing in l, then, since $\lambda > \lambda_c$, the first equation in (7) has a unique solution $l_* > k/(k-1)$. The value of λ_2^* is obtained simply by plugging l_* in the second equation of (6) or (7). Uniqueness of (l_*, λ_2^*) is clear from (7). Also, notice that $l_* > k/(k-1)$ implies $0 < \lambda_2^* < 1$.

We consider now a blocking model with the same parameter λ and parameter $\lambda_2 \in (0, \lambda_2^*)$. Let l be the corresponding fixed point. The fixed point equation applied to l and l_* yields

$$\lambda = \frac{l^k(l-1)}{1-\lambda_2 l} = \frac{l_*^k(l_*-1)}{1-\lambda_2^* l_*}.$$

Also, since $l^k(l-1)/(1-\lambda_2 l)$ is increasing in l and λ_2, inequality $\lambda_2 < \lambda_2^*$ implies

$$\frac{l_*^k(l_*-1)}{1-\lambda_2 l_*} < \frac{l_*^k(l_*-1)}{1-\lambda_2^* l_*} = \frac{l^k(l-1)}{1-\lambda_2 l},$$

which yields $l_* < l$. Finally, notice that $l^k(k-1-k/l)$ increases with l, therefore $\lambda_2 < \lambda_2^*$ and $l_* < l$ imply

$$\lambda\lambda_2 < \lambda\lambda_2^* = l_*^k(k-1-k/l_*) < l^k(k-1-k/l).$$

This inequality is equivalent to $\lambda\lambda_2 + l^k < kl^{k-1}(l-1)$ which, using (4), implies $|f'(l)| > 1$, so l is a repelling fixed point.

Table 1 Critical values

$k=2$		$k=3$		$k=4$	
λ	λ_2^*	λ	λ_2^*	λ	λ_2^*
4.0	0.0	1.69	0.0	1.05	0.00
4.8	0.03	2.03	0.06	1.26	0.08
5.6	0.05	2.36	0.09	1.47	0.13
6.4	0.06	2.70	0.12	1.69	0.17
7.2	0.07	3.04	0.14	1.90	0.20
8.0	0.08	3.38	0.16	2.11	0.22

(iv) If $\lambda \le \lambda_c$ we have $|f'(l)| \le 1$, so the fixed point l is attracting or neutral and it is not obvious that the sequence of iterates $l_{n+1} = f(l_n)$ should converge to l. Noticing that $f(f(l_{odd})) = l_{odd}$ and $f(f(l_{even})) = l_{even}$, we show below that if $\lambda \le \lambda_c$ there can only be a unique root of $f(f(x)) - x$ and so, $l_{odd} = l = l_{even}$.

Suppose a, b are such that $f(a) = b$ and $f(b) = a$. Then, using the first form of the derivative in (4) and then the second, we obtain

$$f'(a)f'(b) = \frac{k^2 a^{k-1} b^{k-1}(a-1)^2}{(\lambda\lambda_2 + a^k)^2} \cdot \frac{b}{b} = \frac{k^2 a^{k-1}(a-1)^2\lambda}{(\lambda\lambda_2 + a^k)(\lambda + a^k)} \left(\frac{(1-\lambda_2)}{a-1} - \lambda_2 \right),$$

so that

$$f'(a)f'(b) < \frac{k^2(a-1)\lambda}{a(\lambda + a^k)}, \tag{8}$$

for $\lambda_2 > 0$. The maximum of the right hand side of (8) occurs at x such that $\lambda = x^k(kx - (k+1))$. Substituting back gives

$$f'(a)f'(b) < \frac{k^2(x-1)x^k(kx-(k+1))}{kx^{k+1}(x-1)} = k\left(k - \frac{k+1}{x}\right) < 1,$$

for $x < k/(k-1)$. If $\lambda = k^k/(k-1)^{k+1}$ then $x = k/(k-1)$, so, since $d\lambda/dx > 0$, for $x > 1$ in the above expression, $\lambda < k^k/(k-1)^{k+1}$ implies $x < k/(k-1)$. Thus the derivative of $f(f(x)) - x$ is negative at all zeros and so there can be at most one of them and therefore, $l_{odd} = l = l_{even}$.

Finally, at $\lambda = \lambda_c$ and $\lambda_2 = 0$ we have

$$f(l_c) = \frac{\lambda_c + l_c}{l_c^k} = 1 + \frac{1}{k-1} = l_c.$$

□

The values of λ_2^* are shown in Table 1 for different λs, starting with $\lambda_c = k^k/(k-1)^{k+1}$.

2.3 The Convergence of the Probability at the Root on $T_k^{(n)}$

We define $T_k^{(n)}$ to be the tree with radius n. It differs from $R_k^{(n)}$ in that the root has $k+1$ edges rather than k. Now all vertices have $k+1$ edges except for those on the boundary which have 1. Calling the respective generating functions for this tree $P_n^T(s,t)$ and $Q_n^T(s,t)$, we have (omitting the s,t variables for the sake of brevity)

$$P_{n+1}^T = \lambda(\lambda_2 P_n + Q_n)^{k+1} \quad \text{and} \quad Q_{n+1}^T = (P_n + Q_n)^{k+1}. \tag{9}$$

Theorem 5 *Let $p_0^{(n)}$ be the probability the central vertex is occupied on $T_k^{(n)}$. Defining l_n by the recursion*

$$l_n = f(l_{n-1}) = \frac{\lambda + l_{n-1}^k}{\lambda\lambda_2 + l_{n-1}^k}, \qquad l_0 = f(1),$$

we have

(i)

$$p_0^{(n)} = \frac{l_n - 1}{l_n + l_{n-1} - 1 - \lambda_2 l_n l_{n-1}}.$$

(ii) *If $\lambda < \lambda_c$ then*

$$p_0^{(n)} \to \frac{l-1}{2l - 1 - \lambda_2 l^2},$$

where l is the unique solution of the equation $l = f(l)$.

(iii) *If $\lambda > \lambda_c$ and $\lambda_2 < \lambda_2^*$ there is a phase-transition in that $l_{2n} \to l_{even}$, $l_{2n+1} \to l_{odd}$, where $l_{odd} < l < l_{even}$, and*

$$\frac{p_0^{(2n)}}{p_0^{(2n+1)}} \to \frac{l_{even} - 1}{l_{odd} - 1}.$$

Proof From (2) and (9)

$$p_0^{(n)} = \frac{P_n^T}{P_n^T + Q_n^T} = \frac{\lambda}{\lambda + l_{n-1}^{k+1}}.$$

Using the recursion we have $\lambda = (l_n - 1)l_{n-1}^k/(1 - l_n\lambda_2)$ and (i) follows. For (ii) and (iii) we take limits. □

We shall see that when the phase-transition occurs, the ratio of occupation probabilities for sites $2n$ steps from the boundary to those for sites $2n+1$ steps from the boundary tends to $(l_{even} - 1)/(l_{odd} - 1)$. The pattern will then be one of alternating rings of higher and lower density.

2.4 The Convergence of the Probability at a Site on $T_k^{(n)}$

We calculate the probability that a site distance m from the root, say s_m, is occupied, $m < n$. The path to s_m is defined as the sequence of sites $s_0, s_1, \dots, s_m$, distances $0, 1, \dots, m$ from the root respectively. We designate by s_{m+1} one of the neighbours of s_m, distance $m+1$ from the root.

It is easy to see that the generating function for the subtree rooted at s_{m+1} is P_{n-m-1}, if s_{m+1} is occupied, and Q_{n-m-1}, when s_{m+1} is unoccupied. Also, if U_0 designates the generating function corresponding to $T_k^{(n)}$, when the subtree rooted at s_1 is excluded and the root is unoccupied, then

$$U_0 = (P_{n-1} + Q_{n-1})^k.$$

If V_0 designates the corresponding generating function when the root is occupied, we have

$$V_0 = \lambda(\lambda_2 P_{n-1} + Q_{n-1})^k.$$

More generally, if U_m is the generating function for $T_k^{(n)}$ when the subtree rooted at s_{m+1} is excluded and s_m is unoccupied, we have

$$U_m = (P_{n-m-1} + Q_{n-m-1})^{k-1}(U_{m-1} + V_{m-1}). \tag{10}$$

When s_m is occupied, the corresponding generating function is given by

$$V_m = \lambda(\lambda_2 P_{n-m-1} + Q_{n-m-1})^{k-1}(U_{m-1} + \lambda_2 V_{m-1}).$$

Finally, if U_m^* and V_m^* designate the generating functions when the subtree rooted at s_{m+1} is included, we have

$$\begin{aligned} U_m^* &= (P_{n-m-1} + Q_{n-m-1})^k (U_{m-1} + V_{m-1}) \\ &= (P_{n-m-1} + Q_{n-m-1}) U_m \end{aligned}$$

and

$$\begin{aligned} V_m^* &= (\lambda_2 P_{n-m-1} + Q_{n-m-1})^k (U_{m-1} + \lambda_2 V_{m-1}) \\ &= (\lambda_2 P_{n-m-1} + Q_{n-m-1}) V_m. \end{aligned}$$

Putting

$$W_m = \frac{U_m + V_m}{U_m + \lambda_2 V_m} \tag{11}$$

gives

$$W_0 = \frac{l_{n-1}^k + \lambda}{l_{n-1}^k + \lambda\lambda_2} = f(l_{n-1}) = l_n \tag{12}$$

and

$$W_m = \frac{l_{n-m-1}^{k-1} W_{m-1} + \lambda}{l_{n-m-1}^{k-1} W_{m-1} + \lambda\lambda_2}. \tag{13}$$

Theorem 6 *Let $p_m^{(n)}$ be the probability that a site distance m from the root is occupied. Then,*

(i) *if $\lambda \le \lambda_c$,*

$$p_m^{(n)} \to \frac{l-1}{2l-1-\lambda_2 l^2}, \tag{14}$$

as $n \to \infty$.

(ii) *If $\lambda > \lambda_c$ and $\lambda_2 < \lambda_2^*$*

$$p_m^{(n)} \to \frac{l_{even} - 1}{l_{even} + l_{odd} - 1 - \lambda_2 l_{even} l_{odd}}, \tag{15}$$

as $n - m \to \infty$ through even values.

(iii) *If* $\lambda > \lambda_c$ *and* $\lambda_2 < \lambda_2^*$

$$p_m^{(n)} \to \frac{l_{odd} - 1}{l_{even} + l_{odd} - 1 - \lambda_2 l_{even} l_{odd}}, \tag{16}$$

as $n - m \to \infty$ *through odd values.*

Proof We use an inductive argument to show that, if $\lambda_2 < \lambda_2^*$ and $\lambda > \lambda_c$, W_m defined in (11) converges either to l_{even} or l_{odd}, as $n - m \to \infty$, through even or odd values, for any $m \geq 0$. The result for $m = 0$ follows from (12) and Lemma 4(iii). Suppose W_{m-1} converges to l_{even} or l_{odd} as $n - (m - 1) \to \infty$ through even or odd values. Then, from (13), if $n - m \to \infty$ through even values, $W_{m-1} \to l_{odd}$ and

$$W_m \to \frac{l_{odd}^{k-1} l_{odd} + \lambda}{l_{odd}^{k-1} l_{odd} + \lambda\lambda_2} = f(l_{odd}) = l_{even}. \tag{17}$$

Analogously, if $n - m \to \infty$ through odd values, $W_m \to l_{odd}$.

On the other hand, if $\lambda \leq \lambda_c$, we have from Lemma 4(iv), $l = l_{odd} = l_{even}$ so that $W_m \to l$, as $n \to \infty$, for all $m \geq 0$.

Notice that

$$p_m^{(n)} = \frac{V_m^*}{U_m^* + V_m^*} = \frac{V_m}{U_m l_{n-m-1} + V_m} = \frac{V_m/U_m}{l_{n-m-1} + V_m/U_m}. \tag{18}$$

Also, from (11), $V_m/U_m = (W_m - 1)/(1 - \lambda_2 W_m)$. Replacing in (18) we obtain

$$p_m^{(n)} = \frac{W_m - 1}{W_m - 1 + (1 - \lambda_2 W_m) l_{n-m-1}}. \tag{19}$$

(i) As $n \to \infty$, l_{n-m-1} and W_m converge to l, so, using (19), convergence (14) follows. For (ii) and (iii), using again (19) and letting $n - m \to \infty$ through even or odd values, (15) and (16) follow. □

2.5 The Convergence of the Probability of a Pattern on $T_k^{(n)}$

We shall now show that the probability of any pattern has a limit as $n \to \infty$ through even values and a possibly different limit as $n \to \infty$ through odd values.

We consider a set of sites A in $T_k^{(n)}$, which remains fixed as n grows. We recall that a pattern $\pi(A)$ is defined by a function from A to $\{0, 1\}$. Let $T_k^{(m)}$ denote the subtree of $T_k^{(n)}$ consisting of all sites distance $\leq m$ from the root. Clearly, for any A, there exists m such that A is contained in $T_k^{(m)}$ so the probability of a pattern $\pi(A)$ can be obtained by adding the probabilities of patterns $\pi(T_k^{(m)})$ whose values on A coincide with those of $\pi(A)$. Hence, the asymptotic behavior of $P^{(2n)}(\pi(A))$ follows from that of $P^{(2n)}(\pi(T_k^{(m)}))$.

Next we prove two technical lemmas.

Lemma 7 *If* $r_0 + r_1 = k^m$, *then*

$$\frac{Q_n^{r_0} P_n^{r_1}}{Q_{n+m} + P_{n+m}} \tag{20}$$

converges, as $n \to \infty$ *through even or odd values.*

Proof For $m=1$, dividing top and bottom by Q_n^k in (20) gives

$$\frac{(P_n/Q_n)^{r_1}}{Q_{n+1}/Q_n^k + P_{n+1}/Q_n^k}. \tag{21}$$

From (1) and (2) we obtain

$$\frac{P_n}{Q_n} = \frac{l_n - 1}{1-\lambda_2 l_n}, \qquad \frac{Q_{n+1}}{Q_n^k} = \frac{(P_n+Q_n)^k}{Q_n^k} = \left[\frac{l_n(1-\lambda_2)}{1-\lambda_2 l_n}\right]^k. \tag{22}$$

Hence, by Lemma 4, (21) converges to possibly different values, as $n\to\infty$ through even or odd values.

We reason inductively to show the assertion is true for all m. Assume it is true for m then, for $r_0+r_1=k^{m+1}$, there exist a,b,c,d such that $r_0 = ak^m+b, r_1 = ck^m+d$, with $b+d=k^m$ and $a+c=k-1$, giving

$$\frac{Q_n^{r_0}P_n^{r_1}}{Q_{n+m+1}+P_{n+m+1}} = \left[\frac{Q_n^{k^m}}{Q_{n+m}+P_{n+m}}\right]^a \left[\frac{P_n^{k^m}}{Q_{n+m}+P_{n+m}}\right]^c$$
$$\times\left[\frac{Q_n^b P_n^d}{Q_{n+m}+P_{n+m}}\right]\left[\frac{(Q_{n+m}+P_{n+m})^k}{Q_{n+m+1}+P_{n+m+1}}\right].$$

The first three brackets have limits by the induction hypothesis and the last, from (1) and Lemma 4. □

Lemma 8 *If* $r_0+r_1=(k+1)k^{m-1}$, *then*

$$\frac{Q_n^{r_0}P_n^{r_1}}{Q_{n+m}^T + P_{n+m}^T}$$

converges, as $n\to\infty$ *through even or odd values.*

Proof Let f_0, g_0, f_1 and g_1 be such that $f_0+g_0=r_0$, $f_1+g_1=r_1$, $f_0+f_1=k^m$ and $g_0+g_1=k^{m-1}$ then,

$$\frac{Q_n^{r_0}P_n^{r_1}}{Q_{n+m}^T+P_{n+m}^T} = \left[\frac{Q_n^{f_0}P_n^{f_1}}{Q_{n+m}+P_{n+m}}\right]\left[\frac{Q_n^{g_0}P_n^{g_1}}{Q_{n+m-1}+P_{n+m-1}}\right]$$
$$\times\left[\frac{(Q_{n+m}+P_{n+m})(Q_{n+m-1}+P_{n+m-1})}{Q_{n+m}^T+P_{n+m}^T}\right].$$

The first two terms on the right converge by Lemma 7. From (1) and (9), the last term can be written as

$$\frac{Q_{n+m}}{Q_{n+m-1}^k}\cdot\frac{(1+P_{n+m}/Q_{n+m})(1+P_{n+m-1}/Q_{n+m-1})}{\lambda(1+\lambda_2 P_{n+m-1}/Q_{n+m-1})^{k+1}+(1+P_{n+m-1}/Q_{n+m-1})^{k+1}}$$

and convergence follows from (22) and Lemma 4. □

Theorem 9 *Let* $\pi(A)$ *be a pattern associated to a fixed set of vertices* A *in* $T_k^{(n)}$, *then*

$$\lim_{n\to\infty} P^{(2n)}(\pi(A)) \quad \textit{and} \quad \lim_{n\to\infty} P^{(2n+1)}(\pi(A))$$

exist.

Proof Assume A is contained in $T_k^{(m)}$, for some m, so it is sufficient to prove the theorem for $\pi(T_k^{(m)})$. If there are r_0 0's and r_1 1's on the boundary and if n_1 is the number of internal occupied sites and n_2 the number of occupied pairs of neighbouring sites, at least one of which is internal, then

$$P^{(n)}(\pi(T_k^{(m)})) = \lambda^{n_1}\lambda_2^{n_2}\frac{Q_{n-m}^{r_0}P_{n-m}^{r_1}}{Q_n^T + P_n^T},$$

which, by Lemma 8, tends to a limit as $n\to\infty$ through even or odd values. □

We study the ratio of probabilities of a pattern and its translated image. A translation of $T_k^{(r)}$ through distance m, $m+r\le n$, is a transformation on $T_k^{(n)}$ mapping the root to any site distance m from the root and preserving all neighbourhood relations of $T_k^{(r)}$. The translation of a pattern $\pi(T_k^{(r)})$ is defined as the pattern $\pi(T_k^{(r)}+m)$ induced on the translated tree $T_k^{(r)}+m$. That is, the 0,1 value of any given site on $T_k^{(r)}$ is equal to that of its translated image.

We shall consider a particular pattern $\pi(T_k^{(r)})$ in which all the outer (boundary) sites are unoccupied.

Lemma 10 *If the pattern $\pi(T_k^{(r)})$ has all sites on its boundary unoccupied, then the weight (generating function) of $\pi(T_k^{(r)}+m)$ is given by*

$$\Gamma_m := \lambda^{n_1}\lambda_2^{n_2}Q_{n-m-r}^{k^r}Q_{n+2-m-r}^{(k-1)k^{r-2}}Q_{n+4-m-r}^{(k-1)k^{r-3}}\cdots Q_{n-m+r-2}^{k-1}U_{m-r}, \tag{23}$$

for $m>r$, and

$$\Gamma_m := \lambda^{n_1}\lambda_2^{n_2}Q_{n-m-r}^{k^r}Q_{n+2-m-r}^{(k-1)k^{r-2}}Q_{n+4-m-r}^{(k-1)k^{r-3}}\cdots Q_{n+m-r-2}^{(k-1)k^{r-m}}Q_{n+m-r}^{k^{r-m}}, \tag{24}$$

for $m\le r$, where n_1 is the number of internal occupied sites and n_2 the number of internal occupied pairs of sites.

Proof Let s_0 be the root of $T_k^{(n)}$ and $s_0,\dots,s_m$ the path from s_0 to s_m, which is the root of $T_k^{(r)}+m$. We follow the same line of reasoning of previous subsections to derive the generating functions. We notice here that internal sites contribute if they are occupied. The remaining contribution depends only on the $(k+1)k^r$ boundary sites of $T_k^{(r)}+m$, which have to be considered according to their distances from s_0.

When $m>r$, the number of boundary sites distance $m+r$ from s_0 is k^r since there are k edges emerging from s_m, with k^{r-1} boundary sites each. One step back we are on s_{m-1}, from which $(k-1)$ edges emerge, with k^{r-2} boundary sites each. Hence, there are $(k-1)k^{r-2}$ boundary sites, distance $m-1+r-1=m+r-2$ from s_0. Further back to s_0, we find that $k-1$ edges emerge from s_i, $i<r$, with k^{r-i-1} boundary sites each. So, there are $(k-1)k^{r-i-1}$ boundary sites, distance $m+r-2i$ from s_0. Finally, there is only one boundary site distance $m-r$ from s_0. (We note that $k^r+(k-1)k^{r-2}+\cdots+(k-1)+1=(k+1)k^{r-1}$.)

The contribution of internal sites to the generating function is $\lambda^{n_1}\lambda_2^{n_2}$. The term due to outer sites distance $m+r-2i$ from s_0 is $Q_{n-(m+r-2i)}^{(k-1)k^{r-i-1}}$, $i\le r-1$, since they are unoccupied. Finally, the contribution of the single site distance $m-r$ is U_{m-r} (see (10)), since the situation corresponds to the exclusion of a subtree rooted at s_{m-r+1} and s_{m-r} is unoccupied. Collecting terms we get (23).

The case $m \le r$ is analogous, the difference being that now s_0 is in $T_k^{(r)} + m$. This implies that boundary sites are distance at least $r - m$ from s_0. For $i < m$, the number of boundary sites distance $m + r - 2i$ is, as before, $(k-1)k^{r-i+1}$ but, for $i = m$, it turns out to be k^{r-m}. For the generating function we proceed as above, except that all terms due to boundary sites are of the Q type. Formula (24) is obtained by collecting terms corresponding to inner and outer sites. □

Lemma 11 *Let $\pi(T_k^{(r)})$ be a pattern with b_1 occupied and $b_0 = (k+1)k^{r-1} - b_1$ unoccupied boundary sites. Let $\rho_m = \Gamma_m/\Gamma_{m+1}$ be the ratio of weights as the center moves from m to $m+1$. Then*

(i) $\rho_m \to L^{b_0} M^{b_1}$ *as $n - m - r \to \infty$ through even values, and*
(ii) $\rho_m \to L^{-b_0} M^{-b_1}$ *as $n - m - r \to \infty$ through odd values,*

where

$$L = \frac{1/l_{even} - \lambda_2}{1/l_{odd} - \lambda_2} \quad \textit{and} \quad M = \frac{1 - 1/l_{even}}{1 - 1/l_{odd}}. \tag{25}$$

Proof We consider first the case with $b_1 = 0$ occupied boundary sites. Notice that, for $m > r$,

$$\begin{aligned} \rho_m = {} & \left[\frac{Q_{n-m-r}}{Q^k_{n-m-r-1}}\right]^{k^{r-1}} \left[\frac{Q^k_{n-m-r}}{Q_{n+1-m-r}}\right]^{(k-1)k^{r-2}} \left[\frac{Q^k_{n+2-m-r}}{Q_{n+3-m-r}}\right]^{(k-1)k^{r-3}} \cdots \\ & \times \left[\frac{Q^k_{n-m+r-4}}{Q_{n-m+r-3}}\right]^{(k-1)} Q^{k-1}_{n-m+r-2} \frac{U_{m-r}}{U_{m+1-r}}. \end{aligned} \tag{26}$$

For the first term of ρ_m above we have, from (22) and Lemma 4,

$$\left[\frac{Q_{n-m-r}}{Q^k_{n-m-r-1}}\right]^{k^{r-1}} = \left[\frac{(1-\lambda_2)l_{n-m-r-1}}{1-\lambda_2 l_{n-m-r-1}}\right]^{k^r} \to \left[\frac{(1-\lambda_2)l_{odd}}{1-\lambda_2 l_{odd}}\right]^{k^r}, \tag{27}$$

as $n - m - r \to \infty$ through even values. For intermediate terms

$$\left[\frac{Q^k_{n-(m+r-2i)}}{Q_{n-(m+r-2i)+1}}\right]^{(k-1)k^{r-i-2}} \to \left[\frac{1-\lambda_2 l_{even}}{(1-\lambda_2)l_{even}}\right]^{(k-1)k^{r-i-1}}, \tag{28}$$

as $n - m - r \to \infty$ through even values, $i = 0, \ldots, r-2$. Finally, for the last term we have

$$Q^{k-1}_{n-m+r-2} \frac{U_{m-r}}{U_{m-r+1}} = 1 \Big/ \left[\frac{P_{n-m+r-2}}{Q_{n-m+r-2}} + 1\right]^{k-1} \left[1 + \frac{V_{m-r}}{U_{m-r}}\right].$$

From (22) and Lemma 4, the first bracket above converges to $((1-\lambda_2)l_{even}/(1-\lambda_2 l_{even}))^{k-1}$ as $n - m + r \to \infty$ through even values. For the second we use (11) and (17) to obtain

$$1 + \frac{V_{m-r}}{U_{m-r}} = \frac{(1-\lambda_2)W_{m-r}}{1-\lambda_2 W_{m-r}} \to \frac{(1-\lambda_2)l_{even}}{1-\lambda_2 l_{even}}, \tag{29}$$

as $n - m + r \to \infty$ through even values.

Collecting limits from (27), (28) and (29), we obtain

$$\rho_m \to \left[\frac{(1-\lambda_2)l_{odd}}{1-\lambda_2 l_{odd}}\right]^{k^r} \left[\frac{1-\lambda_2 l_{even}}{(1-\lambda_2)l_{even}}\right]^{k^r} = \left[\frac{l_{odd}}{l_{even}} \cdot \frac{1-\lambda_2 l_{even}}{1-\lambda_2 l_{odd}}\right]^{k^r} = L^{k^r},$$

as $n-m+r\to\infty$ through even values. Now, if we take into account the b_1 occupied boundary sites, some Q or U terms in (26) have to be replaced by P or V terms. To this end, correcting factors are introduced and their limiting behaviour characterized.

When a boundary site distance $m+r-2i$ from the root is occupied, then the correcting factor in ρ_m, for $i\le r-1$, $m>r$, is

$$\frac{P_{n-(m+r-2i)}}{Q_{n-(m+r-2i)}}\cdot\frac{Q_{n-(m+r-2i)-1}}{P_{n-(m+r-2i)-1}}\to\frac{l_{even}-1}{1-\lambda_2 l_{even}}\cdot\frac{1-\lambda_2 l_{odd}}{l_{odd}-1}=ML^{-1}, \tag{30}$$

as $n-m-r\to\infty$ through even values. The limit follows from the first formula in (22).

If the site distance $m-r$ is occupied ($m>r$), the correcting factor is

$$\frac{V_{m-r}}{U_{m-r}}\cdot\frac{U_{m-r+1}}{V_{m-r+1}}\to\frac{l_{even}-1}{1-\lambda_2 l_{even}}\cdot\frac{1-\lambda_2 l_{odd}}{l_{odd}-1}=ML^{-1}, \tag{31}$$

as $n-m-r\to\infty$ through even values. The limit follows from (29).

Results (30) and (31) show that in the limit, the global correcting factor for ρ_m, corresponding to b_1 occupied boundary sites is $(ML^{-1})^{b_1}$ and convergence (i) follows. For (ii) notice that l_{odd} and l_{even} are interchanged when $n-m-r\to\infty$ through odd values.

When $m\le r$, the result is the same since $(k-1)k^{r-1}+\cdots+(k-1)k^{r-m}+k^{r-m}=k^r$. □

Theorem 12 *Consider a pattern $\pi(T_k^{(r)})$ that has b_1 occupied and $b_0=(k+1)k^{r-1}-b_1$ unoccupied sites on its boundary, then, as $n\to\infty$,*

$$\frac{P^{(n)}(\pi(T_k^{(r)}+m))}{P^{(n)}(\pi(T_k^{(r)}))}\to\begin{cases}1, & \text{if } m \text{ is even},\\ R^{-1}, & \text{if } m \text{ is odd and } n-r \text{ takes even values},\\ R, & \text{if } m \text{ is odd and } n-r \text{ takes odd values},\end{cases}$$

where $R=L^{b_0}M^{b_1}$ and L, M are defined in (25).

Further, for fixed A, m_1, m_2,

$$\lim_{n_1\to\infty}P^{(n_1)}(\pi(A)+m_1)=\lim_{n_2\to\infty}P^{(n_2)}(\pi(A)+m_2),$$

when $n_1, n_2\to\infty$ in such a way that n_1-m_1 and n_2-m_2 always have the same parity.

Proof Note that

$$\frac{P^{(n)}(\pi(T_k^{(r)}+m))}{P^{(n)}(\pi(T_k^{(r)}))}=\prod_{j=0}^{m-1}\frac{P^{(n)}(\pi(T_k^{(r)}+j+1))}{P^{(n)}(\pi(T_k^{(r)}+j))}=\prod_{j=0}^{m-1}\rho_j^{-1}.$$

If m is even and $n-r\to\infty$ through even or odd values, $\rho_0^{-1}\cdots\rho_{m-1}^{-1}$ converges either to $(R^{-1}R)\cdots(R^{-1}R)=1$ or $(RR^{-1})\cdots(RR^{-1})=1$. If m is odd and $n-r\to\infty$ through even values, then the limit is $(R^{-1}R)\cdots(R^{-1}R)\lim\rho_{m-1}^{-1}=R^{-1}$. Finally, if m is odd and $n-r\to\infty$ through odd values, the limit is $(R^{-1}R)\cdots(R^{-1}R)\lim\rho_{m-1}^{-1}=R$. The last assertion follows from Theorems 6 and 9. □

The last line of Theorem 12 shows that it is essentially the parity of the distance from the boundary that determines the distribution at the centre.

2.6 The Markov Property

We recall that a configuration η is a function assigning the value 0 or 1 to each site of $T_k^{(n)}$.

Theorem 13 *If a site s_r in $T_k^{(n)}$ is distance r from the boundary, and $s_r, s_{r+1}, s_{r+2}, \dots, s_{r+m}$ form a chain of neighbouring sites with each s_l distance l from the boundary, then, for $\eta_j \in \{0, 1\}$,*

$$P^{(n)}(\eta(s_r) = 1|\eta(s_{r+1}) = \eta_1, \dots, \eta(s_{r+m}) = \eta_m) = P^{(n)}(\eta(s_r) = 1|\eta(s_{r+1}) = \eta_1),$$

with

$$P^{(n)}(\eta(s_r) = 1|\eta(s_{r+1}) = \eta_1) = (1 - \eta_1)\frac{l_r - 1}{(1 - \lambda_2)l_r} + \eta_1 \frac{\lambda_2(l_r - 1)}{1 - \lambda_2}.$$

Proof Observe that the pattern with $\eta(s_r) = 1, \eta(s_{r+1}) = \eta_1, \dots, \eta(s_{r+m}) = \eta_m$ differs from that only requiring $\eta(s_{r+1}) = \eta_1, \dots, \eta(s_{r+m}) = \eta_m$ in that one of the branches from s_{r+1} is no longer free but must start with a 1. Since s_r is distance r from the boundary and $\eta_1 = 0$, then, instead of weight $P_r + Q_r$ for paths from s_r along that branch, we have P_r. So, the ratio of probabilities (or weights) is $P_r/(P_r + Q_r) = (l_r - 1)/(1 - \lambda_2)l_r$. If $\eta_1 = 1$, then, instead of $\lambda_2 P_r + Q_r$ for paths from s_r along that branch, we have $\lambda_2 P_r$ and the ratio of probabilities is $\lambda_2 P_r/(\lambda_2 P_r + Q_r) = \lambda_2(l_r - 1)/(1 - \lambda_2)$. □

2.7 Correlations

We calculate the correlations on $T_k^{(n)}$ between $\eta(s_0)$ and $\eta(s_m)$, where s_0 is the root and s_m a vertex distance m away. Let

$$p_j^{(n)} = P^{(n)}(\eta(s_j) = 1), \quad p_{lj}^{(n)} = P^{(n)}(\eta(s_l) = 1, \eta(s_j) = 1) \quad \text{and}$$

$$\mathrm{corr}^{(n)}(\eta(s_0), \eta(s_m)) = \frac{p_{0m}^{(n)} - p_0^{(n)} p_m^{(n)}}{\sqrt{p_0^{(n)}(1 - p_0^{(n)})}\sqrt{p_m^{(n)}(1 - p_m^{(n)})}} \tag{32}$$

the correlation between $\eta(s_0)$ and $\eta(s_m)$.

Theorem 14 *Consider the blocking process on $T_k^{(n)}$. Then $\mathrm{corr}^{(n)}(\eta(s_0), \eta(s_m))$ converges to r_m, as $n \to \infty$, with*

$$r_m = (-1)^m \left[\frac{(1 - \lambda_2 l_{even})(l_{even} - 1)}{(1 - \lambda_2)l_{even}} \frac{(1 - \lambda_2 l_{odd})(l_{odd} - 1)}{(1 - \lambda_2)l_{odd}}\right]^{m/2}, \tag{33}$$

when $\lambda_2 \le 1$, and

$$r_m = \left[\frac{(\lambda_2 l - 1)(1 - l)}{(\lambda_2 - 1)l}\right]^m,$$

when $\lambda_2 > 1$.

Proof We first consider the case $\lambda_2 < 1$. Define $u_m^{(n)} = P^{(n)}(\eta(s_m) = 1|\eta(s_0) = 1) = p_{0m}^{(n)}/p_0^{(n)}$. Then, from Theorem 13, we have $u_0^{(n)} = 1$ and

$$\begin{aligned} u_m^{(n)} &= P^{(n)}(\eta(s_m)=1|\eta(s_{m-1})=1)u_{m-1}^{(n)} \\ &\quad + P^{(n)}(\eta(s_m)=1|\eta(s_{m-1})=0)(1-u_{m-1}^{(n)}) \\ &= u_{m-1}^{(n)}\frac{\lambda_2(l_{n-m}-1)}{1-\lambda_2} + (1-u_{m-1}^{(n)})\frac{l_{n-m}-1}{(1-\lambda_2)l_{n-m}} \\ &= -u_{m-1}^{(n)}\alpha_{n-m} + \beta_{n-m}, \end{aligned}$$

with

$$\alpha_{n-m} = \frac{(1-\lambda_2 l_{n-m})(l_{n-m}-1)}{(1-\lambda_2)l_{n-m}}, \qquad \beta_{n-m} = \frac{l_{n-m}-1}{(1-\lambda_2)l_{n-m}}.$$

From Theorem 9, $u_m^{(n)}$ has limits (possibly different) as $n \to \infty$ through even/odd values. Let u_m be the limit as $n \to \infty$ through even values. Then, from Lemma 4,

$$u_{2m} = -u_{2m-1}\alpha_{even} + \beta_{even} \quad \text{and} \quad u_{2m-1} = -u_{2m-2}\alpha_{odd} + \beta_{odd},$$

so,

$$u_{2m} = u_{2(m-1)}\alpha_{odd}\alpha_{even} - \beta_{odd}\alpha_{even} + \beta_{even}.$$

The above recurrence can be solved to yield

$$u_{2m} = (1-p_0)\left[\frac{(1-\lambda_2 l_{even})(l_{even}-1)}{(1-\lambda_2)l_{even}}\frac{(1-\lambda_2 l_{odd})(l_{odd}-1)}{(1-\lambda_2)l_{odd}}\right]^m + p_0,$$

where $p_0 := \lim p_0^{(n)} = (l_{even}-1)/(l_{even}+l_{odd}-1-\lambda_2 l_{even}l_{odd})$, as $n \to \infty$ through even values.

Since $p_{2m} := \lim p_{2m}^{(n)} = p_0$, as $n \to \infty$ through even values, (32) yields

$$\begin{aligned} \mathrm{corr}^{(n)}(\eta(s_0), \eta(s_{2m})) &\to \frac{(u_{2m}-p_{2m})p_0}{p_0(1-p_0)} \\ &= \left[\frac{(1-\lambda_2 l_{even})(l_{even}-1)}{(1-\lambda_2)l_{even}}\frac{(1-\lambda_2 l_{odd})(l_{odd}-1)}{(1-\lambda_2)l_{odd}}\right]^m. \end{aligned}$$

In a similar fashion, the above steps can be repeated to obtain

$$\begin{aligned} \mathrm{corr}^{(n)}(\eta(s_0), \eta(s_{2m+1})) &\to \frac{(u_{2m+1}-p_{2m+1})p_0}{\sqrt{p_0(1-p_0)}\sqrt{p_{2m+1}(1-p_{2m+1})}} \\ &= -\left[\frac{(1-\lambda_2 l_{even})(l_{even}-1)}{(1-\lambda_2)l_{even}}\frac{(1-\lambda_2 l_{odd})(l_{odd}-1)}{(1-\lambda_2)l_{odd}}\right]^{m+1/2}, \end{aligned}$$

as $n \to \infty$ through even values.

Combining these results and noticing their symmetry in l_{even}, l_{odd}, so that they do not depend on whether $n \to \infty$ through even or odd values, we obtain (33).

When $\lambda_2 > 1$ all correlations are positive. From (32) and Lemma 4, we have $l_n \to l$ and

$$\mathrm{corr}^{(n)}(\eta(s_0), \eta(s_m)) \to \frac{(u_m-p_m)}{(1-p_0)} = \left[\frac{(\lambda_2 l-1)(1-l)}{(\lambda_2-1)l}\right]^m. \qquad \square$$

We note that the decay of the correlation is geometric whereas in the RSA models, the decay goes as $1/m!$.

3 The Dimer Model

In the dimer model pairs of particles arrive at empty pairs of neighbouring sites at rate λ and are removed at rate 1. The analysis follows that of the blocking process with the same notation.

The probability of a configuration is proportional to $\lambda^{\#\text{dimers}}$. If the root of $R_k^{(n)}$ is not occupied, then there is no restriction on placing dimers on the k subtrees. If the root is occupied, then there must be a dimer occupying the root and one of the k neighbouring vertices. This vertex is adjacent to k rooted trees each of size $n-2$. On the $k-1$ other vertices no restrictions are placed. Thus,

Lemma 15 *For the dimer process on* $R_k^{(n)}$

$$Q_{n+1} = (Q_n + P_n)^k \tag{34}$$

and

$$P_{n+1} = k\lambda(Q_n + P_n)^{k-1}(Q_{n-1} + P_{n-1})^k = k\lambda Q_n(Q_n + P_n)^{k-1}. \tag{35}$$

On $T_k^{(n)}$

$$Q_{n+1}^T = (Q_n + P_n)^{k+1} \quad \text{and} \quad P_{n+1}^T = (k+1)\lambda Q_n(Q_n + P_n)^k. \tag{36}$$

Lemma 16 *Let*

$$f(x) = \frac{k\lambda}{1+x}, \quad x \ge 0, \quad \text{and} \quad l_n = \frac{P_n}{Q_n},$$

where P_n, Q_n *are defined in* (34) *and* (35). *Then*

(i) l_n *satisfies the recursion* $l_{n+1} = f(l_n)$.
(ii) *The sequence* (l_n) *converges to* $l < 1$, *the unique solution of* $f(l) = l$.

Proof Assertion (i) follows from $l_{n+1} = k\lambda Q_n(Q_n+P_n)^{k-1}/(Q_n+P_n)^k = k\lambda/(1+l_n)$. For (ii) notice that, since f is continuous and decreasing, the fixed point theorem implies that $f(x) = x$ has a unique solution l. Besides, $|f'(x)| = k\lambda/(1+x)^2$ and $|f'(l)| = l/(1+l) < 1$ so l is an attracting fixed point. Also, since $|f'(x)| < 1$ for $x > \sqrt{k\lambda} - 1$ and $l_0 = 0, l_1 = f(l_0) = k\lambda > \sqrt{k\lambda} - 1$, the sequence (l_n) converges to the fixed point l. □

3.1 The Convergence of the Probability at the Root on $T_k^{(n)}$

Theorem 17 *Let* $p_0^{(n)}$ *be the probability the central vertex is occupied on* $T_k^{(n)}$. *Then*

$$p_0^{(n)} = \frac{(k+1)l_n}{k+(k+1)l_n} \to \frac{(k+1)l}{k+(k+1)l},$$

as $n \to \infty$, *where* $l = (\sqrt{1+4k\lambda} - 1)/2$.

Proof From (36) we have

$$p_0^{(n)} = \frac{P_n^T}{Q_n^T + P_n^T} = \frac{(k+1)\lambda Q_{n-1}(Q_{n-1}+P_{n-1})^k}{(Q_{n-1}+P_{n-1})^{k+1} + (k+1)\lambda Q_{n-1}(Q_{n-1}+P_{n-1})^k}$$
$$= \frac{\lambda(k+1)}{1 + l_{n-1} + \lambda(k+1)} = \frac{(k+1)l_n}{k+(k+1)l_n}.$$

Convergence follows from Lemma 16(ii) and the value of l is obtained as the unique positive solution of equation $l^2 + l - k\lambda = 0$. □

3.2 The Convergence of the Probability at a Site on $T_k^{(n)}$

Because (l_n) has a unique limit point many of the complications of the blocking process do not arise for the dimer model. However, some difficulties emerge owing to the dimers occupying pairs of sites. As in Sect. 2.4, we consider a site s_m distance m from the root. The path to s_m is the sequence of sites $s_0, s_1, \ldots, s_m$, distances $0, 1, \ldots, m$ from the root respectively. We designate by s_{m+1} one of the neighbours of s_m, distance $m+1$ from the root.

We consider several generating functions and give recurrence relations among them. Let A_m be the generating function when the subtree rooted at s_{m+1} is excluded and s_m is unoccupied; B_m is the generating function when the subtree rooted at s_{m+1} is excluded and there is a dimer at s_{m-1}, s_m; C_m is as B_m but the dimer is at s_m, s_{m+1}. Finally, let D_m be as C_m but the dimer is at $s_m, \tilde{s}_{m+1}$, where $\tilde{s}_{m+1} \neq s_{m+1}$ is another site distance $m+1$ from the root.

Putting $S_m = P_m + Q_m$, we then have the following recurrence relations:

$$\begin{aligned} A_m &= S_{n-m-1}^{k-1}(A_{m-1} + B_{m-1} + D_{m-1}), \\ B_m &= S_{n-m-1}^{k-1} C_{m-1}, \\ C_m &= S_{n-m-1}^{k-1} \lambda (A_{m-1} + B_{m-1} + D_{m-1}), \\ D_m &= S_{n-m-1}^{k-2} Q_{n-m-1}(k-1)\lambda(A_{m-1} + B_{m-1} + D_{m-1}). \end{aligned} \tag{37}$$

Further, using a superscript $*$ to indicate the generating functions when s_{m+1} is included, we have:

$$\begin{aligned} A_m^* &= A_m S_{n-m-1}, \qquad & B_m^* &= B_m S_{n-m-1}, \\ C_m^* &= C_m Q_{n-m-1}, \qquad & D_m^* &= D_m S_{n-m-1}. \end{aligned} \tag{38}$$

Theorem 18 *Let $p_m^{(n)}$ be the probability that s_m is occupied and $q_m^{(n)}$ the probability of a dimer at s_m, s_{m+1}. Then*

$$p_m^{(n)} \to \frac{(k+1)l}{k + (k+1)l}, \quad \text{and} \quad q_m^{(n)} \to \frac{l}{k + (k+1)l},$$

where $l = (\sqrt{1 + 4k\lambda} - 1)/2$.

Proof From Lemma 16 and relations (37, 38), we have $D_m^* = (k-1)C_m^*$ and

$$\frac{C_m^* + D_m^*}{A_m^*} = k \frac{C_m Q_{n-m-1}}{A_m S_{n-m-1}} = \frac{k\lambda}{1 + l_{n-m-1}} = l_{n-m}. \tag{39}$$

Also,

$$\begin{aligned} \frac{B_m^*}{A_m^*} = \frac{B_m}{A_m} &= \frac{C_{m-1}}{A_{m-1} + B_{m-1} + D_{m-1}} \\ &= \frac{\lambda}{1 + B_{m-1}/A_{m-1} + (k-1)\lambda/(1 + l_{n-m-1})} \end{aligned} \tag{40}$$

and

$$\frac{B_1}{A_1} = \frac{\lambda S_{n-1}^k}{S_{n-1}^k + k\lambda Q_{n-1} S_{n-1}^{k-1}}$$
$$= \frac{\lambda}{1 + k\lambda/(1+l_{n-1})} = \frac{\lambda}{1+l_n} = \frac{l_{n+1}}{k} \to \frac{l}{k}, \tag{41}$$

as $n \to \infty$. Using (40) and (41), an inductive argument shows that

$$\frac{B_m}{A_m} \to \frac{l}{k}, \tag{42}$$

as $n \to \infty$. Finally, from (39), (40) and (42), we have

$$p_m^{(n)} = \frac{B_m^* + C_m^* + D_m^*}{A_m^* + B_m^* + C_m^* + D_m^*}$$
$$= \frac{B_m^*/A_m^* + l_{n-m}}{1 + B_m^*/A_m^* + l_{n-m}} \to \frac{(k+1)l}{k + (k+1)l},$$

as $n \to \infty$, obtaining the same asymptotic probability as at the root.

Similarly, for the probability of a dimer at s_m, s_{m+1}, we have

$$q_m^{(n)} = \frac{C_m^*}{A_m^* + B_m^* + C_m^* + D_m^*} \to \frac{l/k}{1 + l/k + l} = \frac{l}{k + (k+1)l},$$

which is $1/(k+1)$ times the probability s_m is occupied as would be expected. □

3.3 The Convergence of the Probability of a Pattern on $T_k^{(n)}$

The rest closely follows Sect. 2.4. We shall indicate where the treatment differs. Lemmas 7 and 8 and Theorem 9 remain the same. Equation (23) in Lemma 10 has A_{m-r} in place of U_{m-r}. In expression (26) for ρ_m, all terms except the last one remain the same. It is

$$Q_{n-m+r-2}^{k-1} \frac{A_{m-r}}{A_{m+1-r}} = \frac{Q_{n-m+r-2}^{k-1}}{S_{n-m+r-2}^{k-1}} \frac{A_{m-r}}{A_{m-r} + B_{m-r} + D_{m-r}} \tag{43}$$
$$\to \frac{1}{(1+l)^{k-1}} \frac{1}{1 + \frac{\lambda}{(1+l)} + \frac{(k-1)\lambda}{(1+l)}} = \frac{1}{(1+l)^k},$$

using (37) and (42).

If we now substitute for 0s on the boundary 1s which belong to dimers which lie within $T_k^{(r)}$, then the only change to (43) is substituting C_{m-r} in place of A_{m-r}. Since $C_m/A_m = C_{m+1}/A_{m+1} = \lambda$, the result is not changed. Similarly, if we now substitute for 0s on the boundary 1s which do not belong to dimers which lie within $T_k^{(r)}$, then we substitute terms like P_{n-m-r} for Q_{n-m-r}, but, since in the limit $P_{n-m-r}/Q_{n-m-r} = P_{n-m+1-r}/Q_{n-m+1-r}$ and $B_m/A_m = B_{m+1}/A_{m+1}$, the ratio does not change when the pattern is displaced.

Theorem 19 *In the dimer model, if $\pi(A)$ is a pattern fixed relative to the root then, for all m,*

$$\lim_{n\to\infty} P^{(n)}(\pi(A)) = \lim_{n\to\infty} P^{(n)}(\pi(A) + m).$$

A process which superficially looks like the dimer model is the Double Flipping Process (DFP) in which the only flips allowed are $11 \to 00$ at rate b and $00 \to 11$ at rate a. It is shown in [9] that, if the initial measure is translation invariant, the process converges to the product measure with density $\sqrt{a}/(\sqrt{a}+\sqrt{b})$. The difference can be seen in the following transitions $0000 \to 0011 \to 1111 \to 1001$, allowable in the DFP but not in the dimer model.

3.4 Correlations

We calculate the correlation between $\eta(s_0), \eta(s_m)$ in the limit, as $n \to \infty$, using the same notation as in Sect. 2.6 for the blocking process. We condition on $\eta(s_0) = 0$. There are then three possibilities for s_1 along the path leading from s_0 to s_m. It can be unoccupied ($\eta(s_1) = 0$) or it can be occupied by one end of a dimer along the path to s_m or it can be occupied by one end of a dimer not along the path to s_m, both of the latter possibilities having $\eta(s_1) = 1$. In general if there is a 0 at position s_r, then the generating function for the subtree rooted at s_{r+1} is $P_{n-r-1} + Q_{n-r-1}$, whereas if there is also a 0 at s_{r+1} it is Q_{n-r-1}, so that $P^{(n)}(\eta(s_{r+1}) = 0|\eta(s_r) = 0) = Q_{n-r-1}/(P_{n-r-1} + Q_{n-r-1}) \to 1/(l+1)$, as $n \to \infty$. Given $\eta(s_{r+1}) = 1$, the probability is $1/k$ that the dimer lies along the path to s_m, $(k-1)/k$ that it does not. These conditional probabilities are the same if $\eta(s_r) = 1$ and the dimer which covers s_r does not cover s_{r+1}. Thus, as we move from the root to s_m, we have a regeneration point wherever there is either a 0, or one end of a dimer with the other end not on the path, or the second end of a dimer which lies on the path.

Theorem 20 *For the dimer model on* $T_k^{(n)}$, $\mathrm{corr}^{(n)}(\eta(s_0), \eta(s_m))$ *converges to* r_m, *as* $n \to \infty$, *with*

$$r_m = \frac{1}{(k+1)(1+l)} \left(\frac{-l}{k(1+l)} \right)^{m-1},$$

for $m \geq 1$.

Proof Let P_0 denote the asymptotic probability measure for the dimer model, conditional on $\eta(s_0) = 0$, that is, $P_0(\cdot) = \lim_{n\to\infty} P^{(n)}(\cdot|\eta(s_0) = 0)$. Let $\pi_r = P_0(\eta(s_r) = 0)$ and $q_r = P_0$ (s_r is a regeneration point). Then, clearly $q_0 = 1$ and

$$\pi_r = \frac{1}{1+l} q_{r-1}, \tag{44}$$

for $r \geq 1$, since the (limiting) probability of having a 0 right after a regeneration point is $1/(1+l)$. Of course, if s_{r-1} is not a regeneration point, $\eta(s_r) = 0$ is impossible.

On the other hand, s_r is not a regeneration point if and only if s_{r-1} is a regeneration point and s_r is the beginning of a dimer lying along the path to s_m. In terms of conditional probabilities we have

$$1 - q_r = \frac{l}{k(1+l)} q_{r-1}. \tag{45}$$

From (44) and (45) we obtain the recursion

$$\pi_r = \frac{1}{1+l} - \frac{l}{k(1+l)} \pi_{r-1},$$

with $\pi_1 = 1/(1+l)$, which is readily solved to yield

$$\pi_r = \frac{k}{k+(k+1)l}\left(1-\left(\frac{-l}{k(1+l)}\right)^r\right),$$

for $r \geq 1$. Finally, from Theorems 17 and 18, and formula (32), we have $p_0 := \lim_{n\to\infty} p_0^{(n)} = \lim_{n\to\infty} p_m^{(n)} = (k+1)l/(k+(k+1)l)$ and $\mathrm{corr}^{(n)}(\eta(s_0), \eta(s_m)) = \mathrm{corr}^{(n)}(1-\eta(s_0), 1-\eta(s_m)) \to r_m$, with

$$\begin{aligned} r_m &= \frac{\pi_m(1-p_0)-(1-p_0)^2}{(1-p_0)p_0} \\ &= \frac{\pi_m-(1-p_0)}{p_0} = -\frac{k}{(k+1)l}\left(\frac{-l}{k(1+l)}\right)^m. \end{aligned}$$

□

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